SOLUTION OF NONLINEAR PROBLEMS OF THE THEORY OF HEAT CONDUCTION BY THE KANTOROVICH METHOD

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The Kantorovich method is used to obtain an approximate solution of the problem of heating (cooling) of a plate, whose thermophysical parameters depend on temperature, in the presence of radiative heat exchange with the ambient medium. The solution is compared with certain known exact solutions.

We shall consider the symmetrical heating (or cooling) of a plate whose thermal coefficients depend on temperature in the case of nonlinear boundary conditions. There are no internal heat sources. To simplify the calculations, we introduce the new temperature function

$$\vartheta = \frac{1}{\lambda(T)} \int_0^T \lambda(T)\, dT, \quad \frac{\partial}{\partial x}(\lambda\vartheta) = \lambda \frac{\partial T}{\partial x}. \tag{1}$$

Obviously, if $\lambda = \text{const}$, then ϑ is simply the temperature T.

With the new variable the differential equation of heat conduction and the boundary conditions take the form

$$L(\lambda\vartheta) = \frac{\partial^2}{\partial x^2}(\lambda\vartheta) - \frac{\rho c}{\lambda}\frac{\partial}{\partial \tau}(\lambda\vartheta) = 0,$$

$$0 < x < R, \quad \tau > \tau_{in}; \tag{2}$$

$$M(\lambda\vartheta) = \frac{\partial}{\partial x}(\lambda\vartheta) = \begin{cases} p(\vartheta), & x = R, \quad \tau > \tau_{in} \\ 0, & x = 0, \quad \tau > \tau_{in}; \end{cases} \tag{3}$$

$$\vartheta = \vartheta_{in}, \quad 0 \leqslant x \leqslant R, \quad \tau = \tau_{in}. \tag{4}$$

Following Kantorovich [1] we represent the approximate solution of this problem in the form of a sum

$$\widetilde{\lambda\vartheta} = \sum_{i=1}^{l} f_i(\tau)\, \varphi_i(\tau, x), \tag{5}$$

where f_i are unknown functions, and φ_i given (coordinate) functions selected a priori. We also require that at each given instant $\widetilde{\lambda\vartheta}$ be in a certain sense close to the exact solution $\lambda\vartheta$ in the closed region $0 \le x \le R$. In other words, with respect to the space coordinates, we apply the usual procedure of direct methods, and with respect to the time coordinate the Kantorovich method.

In the paper cited the Kantorovich method was used in combination with the method of moments, which consists in satisfying, together with the boundary condition, the conditions of orthogonality of the residues obtained as a result of substituting $\widetilde{\lambda\vartheta}$ for $\lambda\vartheta$ in (2) with the first l functions of some system $\{\zeta_k\}$:

$$\int_0^R L(\widetilde{\lambda\vartheta})\, \zeta_k\, dx = 0; \quad k = 1, 2, \ldots, l. \tag{6}$$

If, in particular, we take as the functions ζ_k the coordinate functions φ_i, the method of moments goes over into the Galerkin method.

The applicability of the Galerkin method for solving linear problems was established in [2], and the convergence of the method of moments in solving nonlinear problems was proved in [3].

After performing the integration in (6) we obtain a system of differential equations (generally nonlinear) for determining the unknown functions of time f_i. The solution of a high-order system is difficult and therefore in (5) we retain only the first two terms. In order not to lose accuracy, we introduce additional physical considerations and select the coordinate functions in accordance with the special characteristics of the problem.* We will consider two stages of the process and assume that in the first stage the heating zones spread from both surfaces of the plate into its interior, the inner layer (still unaffected by heating) retaining the initial temperature (Fig. 1). When the two fronts come together in the middle plane, the second stage begins and the heating zone embraces the entire body.

On the basis of the theory of the quasi-stationary regime [4] we assume that at any given moment the divergence of the heat flux density is constant within the heated zone and, consequently, a flux of density p passing through the surface is uniformly distributed in that zone. This assumption may be regarded as a macroscopic analogy of the condition of local equilibrium of the microscopic parts of a system:

$$\frac{\partial}{\partial x}\left(\lambda \frac{\partial \tilde{T}}{\partial x}\right) = \frac{\partial^2}{\partial x^2}(\widetilde{\lambda\vartheta}) = \frac{p}{q}. \tag{7}$$

In the next instant, of course, the value of the divergence changes since the energy flux through the surface will already be different (and in the first stage of the process, moreover, the width q of the heating zone will increase).

*Thus, it is possible to obtain a solution perfectly suitable for engineering calculations. This is illustrated later by comparing certain of the equations derived with known exact expressions. Good agreement was also obtained when the approximate solutions were compared with the results of analog simulation and computer calculations.

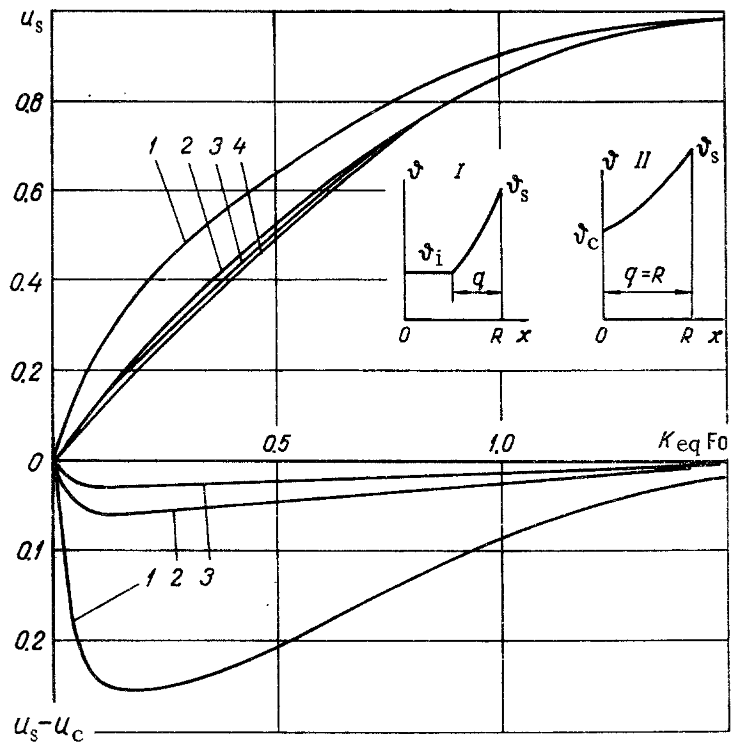

Fig. 1. Variation of relative surface temperature u_s and temperature drop $u_s - u_c$ over cross section for a plate heated by radiative heat transfer from an external medium (initial temperature of plate zero): I and II) first and second stages; 1) at K_{eq} = 0.50, 2) 0.10; 3)0.05; 4) 0.

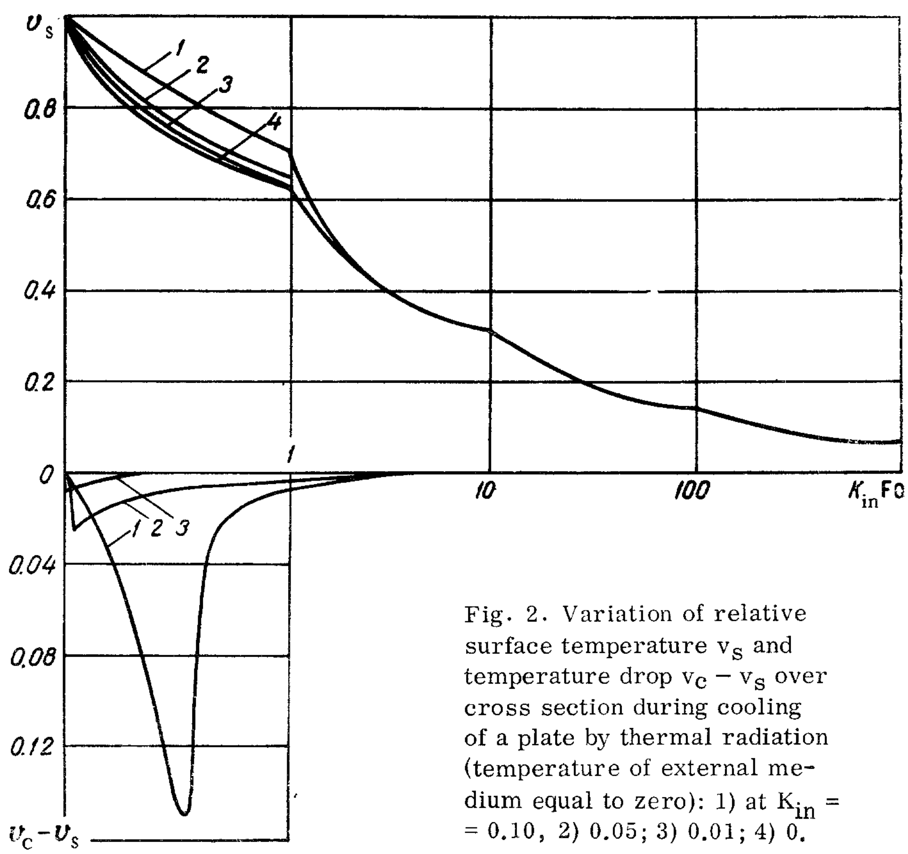

Fig. 2. Variation of relative surface temperature v_s and temperature drop $v_c - v_s$ over cross section during cooling of a plate by thermal radiation (temperature of external medium equal to zero): 1) at K_{in} = = 0.10, 2) 0.05; 3) 0.01; 4) 0.

After integrating (7) with boundary conditions (3) we obtain for the first stage ($q < R$)

$$(\widetilde{\lambda\vartheta}) = \begin{cases} 1\cdot(\lambda\vartheta)_{in} + pq\cdot\dfrac{1}{2}\left(1-\dfrac{R-x}{q}\right)^2, & R-q<x<R \\ 1\cdot(\lambda\vartheta)_{in}, & 0<x<R-q. \end{cases} \quad (8)$$

In the second stage $q = R$ and solution (8) takes the form

$$(\widetilde{\lambda\vartheta}) = (\widetilde{\lambda\vartheta})_c\cdot 1 + pq\cdot\frac{1}{2}\left(\frac{x}{R}\right)^2,\ 0<x<R. \quad (9)$$

Here, in accordance with the form of expression (5) the first factors are the functions f_i, and the second the coordinate functions φ_i.

We now apply the method of moments with respect to the x coordinate, setting

$$\zeta_i = \frac{\partial}{\partial q}\varphi_i \quad (10)$$

in (6). It follows from (8) and (9) that $\zeta_1 = 0$, and in the first stage ($q < R$)

$$\zeta_2 = \begin{cases} \left(1-\dfrac{R-x}{q}\right)\dfrac{R-x}{q^2}, & R-q<x<R \\ 0 & 0<x<R-q \end{cases}, \quad (10a)$$

while in the second stage ($q = R$)

$$\zeta_2 = -\frac{1}{R}\left(\frac{x}{R}\right)^2 = -\frac{2}{R}\varphi_2;\ 0<x<R. \quad (10b)$$

Thus, in the second stage the function ζ_2 coincides with the coordinate function φ_2 (correct to a constant factor), and, consequently, the method of moments goes over into the Galerkin method.

From (2), (8), and (9) we obtain the approximate value of the operator $L(\widetilde{\lambda\vartheta})$ in the first stage (for the heated zone)

$$L(\widetilde{\lambda\vartheta}) = \frac{p}{q} - \frac{\rho c}{2\lambda}\left\{\left[1-\left(\frac{R-x}{q}\right)^2\right]p\frac{\partial q}{\partial\tau} + \left(1-\frac{R-x}{q}\right)^2 q\frac{\partial p}{\partial\tau}\right\},\ R-q<x<R \quad (11)$$

and in the second stage

$$L(\widetilde{\lambda\vartheta}) = \frac{p}{q} - \frac{\rho c}{\lambda}\left[\frac{\partial}{\partial\tau}(\lambda\vartheta)_c + \frac{R}{2}\left(\frac{x}{R}\right)^2\frac{\partial p}{\partial\tau}\right],\quad 0<x<R. \quad (11a)$$

Substituting these expressions in (6) and performing the integration, we should obtain a system of ordinary differential equations for determining the unknown time functions. However, since $\zeta_1 = 0$, the system reduces to only one equation. At the same time, (11) and (11a) each contain two unknown functions (p and q or p and $\lambda\vartheta_c$). It is easy to see, however, that they are not independent, since they are related by boundary condition (3). Therefore, in solving the differential equation they can be expressed one in terms of the other or, when this is convenient (as, for example, in the case considered below), in terms of a third related time function.

The specific form of the solution depends strongly on the variation of λ, ρc, and p during heating (or cooling). In order to express the effect of each nonlinearity more clearly, we divide the general problem into two particular ones:

1. Linear differential equation of heat conduction, nonlinear boundary conditions.

2. Nonlinear differential equation, linear boundary conditions.

Heating (cooling) of plate with constant thermal coefficients. The process takes place in an electromagnetic field with a strongly expressed skin effect. There is radiative heat exchange with the surrounding medium. The initial temperature is constant. Since λ = const, it follows from (1) that $\vartheta = T$.

We will start by considering the first stage of heating. We substitute (10a) and (11) in (6). After integration with respect to the x coordinate over the entire cross section of the plate we obtain

$$\frac{1}{6}\frac{p}{q} - \frac{1}{2a}\left(\frac{7}{60}p\frac{\partial q}{\partial\tau} + \frac{1}{20}q\frac{\partial p}{\partial\tau}\right) = 0. \quad (12)$$

We now express p and q in terms of the surface temperature ϑ_s. From boundary condition (3) we have

$$\lambda\frac{\partial\vartheta}{\partial x} = p = \gamma\sigma(\vartheta_{eq}^4 - \vartheta_s^4),\ p_{eq} = \gamma\sigma\vartheta_{eq}^4 = p_{em} + \gamma\sigma\vartheta_c^4, \quad (13)$$

where the equivalent flux density p_{eq} (or the equivalent temperature of the medium ϑ_{eq}) takes into account both transfer of the energy of the electromagnetic field with flux density p_{em} and radiative heat exchange between the surface of the plate at temperature ϑ_s and the outer wall (at temperature ϑ_m). Then, setting $x = R$, from (8) and (13) we find

$$q = \frac{2\lambda}{p}(\vartheta_s - \vartheta_{in}) = \frac{2\lambda}{\gamma\sigma}\frac{\vartheta_s - \vartheta_{in}}{\vartheta_{eq}^4 - \vartheta_s^4}. \quad (14)$$

We now introduce the relative temperature, representing it in fractions of the equivalent temperature of the medium ϑ_{eq} or the initial temperature of the body ϑ_{in} (one of these quantities is not equal to zero), together with the criterion as the Biot number for linear boundary conditions:

$$u = \frac{\vartheta}{\vartheta_{eq}};\ v = \frac{\vartheta}{\vartheta_{in}};\ K_{eq} = (\gamma\sigma\vartheta_{eq}^4)\left(\lambda\frac{\vartheta_{eq}}{R}\right)^{-1};$$
$$K_{in} = (\gamma\sigma\vartheta_{in}^4)\left(\lambda\frac{\vartheta_{in}}{R}\right)^{-1}. \quad (15)$$

Then (14) takes the form

$$\frac{q}{R} = \frac{2}{K_{eq}}\frac{u_s - u_{in}}{1 - u_s^4}. \quad (14a)$$

After substituting (13) and (14) in (12) and integrating with respect to time with initial condition (4), we obtain

$$\int_{z=u_{in}}^{z=u_s}\left[\frac{2}{5}\left(\frac{z-u_{in}}{1-z^4}\right)^2 + \frac{3}{20}\frac{z(z-u_{in})}{1-z^4} + \frac{3}{40}\ln\left|\frac{1+z^2}{1-z^2}\right| - \right.$$

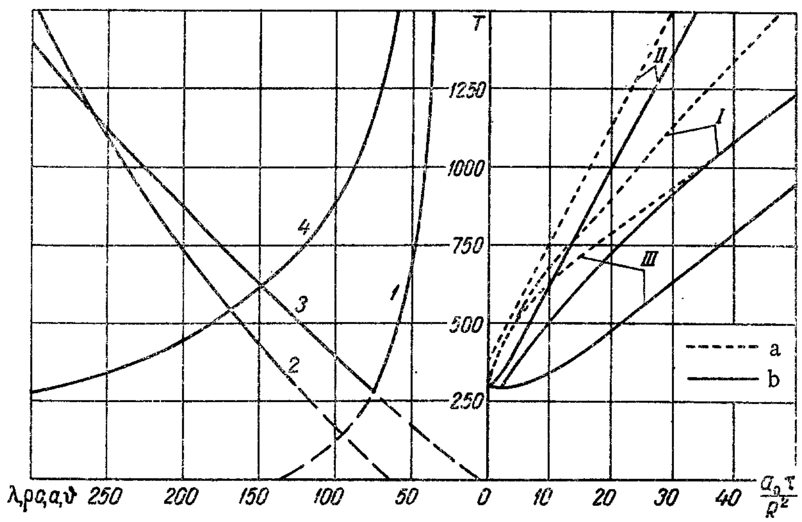

Fig. 3. Variation of temperature T, °K of surface (a) and middle plate (b) of a plate heated by a constant heat flux with allowance for the dependence of the thermophysical parameters (1) λ, W/m · deg, 2) 5 · 10^6 ρc, J/m^3 · deg; 3) 0.1 ϑ, deg: 4) 10^6 a, m^2/sec) on temperature (I), at a =

$$-\frac{9}{40}u_{in}\left(\frac{1}{2}\ln\left|\frac{1+z}{1-z}\right| + \operatorname{arctg} z\right)\Bigg] =$$

$$= K_{eq}^2\text{Fo}, \ \text{Fo} = \frac{\lambda}{\rho c}\frac{\tau}{R^2}. \tag{16}$$

Having determined from (16) the relative surface temperature for a given instant of time, we then calculate from (13) and (14) the resultant flux density p and the depth of the heated zone q and, finally, from (8) find the temperature distribution over the plate cross section.

The first stage ends at $\text{Fo} = \text{Fo}_1$, when the two heating fronts meet. The corresponding relative surface temperature $u_{s,1}$ is determined from (13a) by setting q = R. The temperature distribution attained at that moment is the initial distribution for the second stage. The solution for the second stage is similarly obtained, but using (10b) and (11a):

$$\left[\frac{1}{4}\ln\left|\frac{1+z}{1-z}\right| + \frac{1}{2}\operatorname{arctg} z - \frac{1}{10}\ln|1-z^4|\right]_{z=u_{s,1}}^{z=u_s} =$$

$$= K_{eq}(\text{Fo} - \text{Fo}_1). \tag{17}$$

Having determined the surface temperature at any instant from (17), we can then find from (9) and (13) the relative temperature in the middle plane and at any other intermediate point

$$u_c = u_s - \frac{K_{eq}}{2}(1-u_s^4) \tag{18}$$

Several special cases are of interest.

1. The initial temperature is equal to zero. In this case all the formulas are simplified, since they contain only one parameter K_{eq} (Fig. 1).

2. Heating of a thin plate. In this case K_{eq}, and therefore it is possible to disregard the first stage ($\text{Fo}_1 \approx 0$), assuming on the basis of (14a) and (18) that at the very beginning $u_s \approx u_c = u$. From (17) we have

$$\left[\frac{1}{4}\ln\left|\frac{1+z}{1-z}\right| + \frac{1}{2}\operatorname{arctg} z - \frac{1}{10}\ln|1-z^4|\right]_{z=u_{in}}^{z=u} = \frac{p_{eq}\tau}{\rho c R\,\vartheta_{eq}},$$

which differs from the known exact formula only with respect to the very small third term on the left (the relative error at u = 0.5 is less 0.6%, and at u = 0.9 about 2%).

3. Heating ceases at a low relative temperature when it is possible to neglect the radiation of the plate itself as compared with p_{eq}. This is the linear problem. Setting $u_c = \vartheta_c/\vartheta_{eq} \ll 1$ and going over to absolute temperatures, from (8), (14), and (15) we obtain the solution for the first stage:

$$\tilde{\vartheta} = \vartheta_{in} + \sqrt{\frac{10}{7}}\frac{p_{eq}R}{\lambda}\sqrt{\text{Fo}}\left[1 - \sqrt{\frac{7}{40}}\frac{R-x}{R\sqrt{\text{Fo}}}\right]^2,$$

$$\frac{q}{R} = \sqrt{\frac{40}{7}\text{Fo}}.$$

At the surface the temperature

$$\tilde{\vartheta}_s = \vartheta_{in} + 1.19\frac{p_{eq}R}{\lambda}\sqrt{\text{Fo}}.$$

However, the exact solution for small Fo has the form [4]

$$\vartheta_s = \vartheta_{in} + 2\,\text{ierfc}\,0\cdot\frac{p_{eq}R}{\lambda}\sqrt{\text{Fo}} = \vartheta + 1.13\frac{p_{eq}R}{\lambda}\sqrt{\text{Fo}},$$

i.e., the relative error is about 5.5%.

The first stage ends at $\text{Fo}_1 = 7/40$. The approximate solution for the second stage, obtained from (9) and (18), has the form

$$\tilde{\vartheta} = \vartheta_{in} + \frac{p_{eq}R}{\lambda}\left[\text{Fo} + \frac{1}{4}\left(\frac{x}{R}\right)^2 - \frac{7}{40}\right],$$

and the exact solution for the quasi-stationary regime is

$$\vartheta = \vartheta_{in} + \frac{p_{eq}R}{\lambda}\left[\text{Fo} + \frac{1}{2}\left(\frac{x}{R}\right)^2 - \frac{7}{42}\right].$$

4. The equivalent temperature of the external medium is equal to zero. The plate is cooled from the initial temperature ϑ_{in}. We transform solution (8) and (9), expressing pq in terms of the surface temperature (for this it is first necessary to set x = R in the equations) and introducing the relative temperature v and the criterion K_{in} given by (15). We then pass to the limit $\vartheta_{eq} \to 0$ in the general equations (16) and (17). Finally, for the first stage we obtain

$$\frac{v-1}{v_s-1} = \left(1 - \frac{R-x}{q}\right)^2, \quad \frac{q}{R} = \frac{2}{K_{in}}\frac{1-v_s}{v_s^4},$$

$$\frac{1}{70} + v_s^{-3}\left[\frac{2}{5} - \frac{5}{7}v_s + \frac{3}{10}v_s^2\right] = K_{in}^2\text{Fo}$$

and for the second stage

$$\frac{v-v_c}{v_s-v_c} = \left(\frac{x}{R}\right)^2, \quad v_c = v_s + \frac{K_{in}}{2}v_s^4,$$

$$\frac{1}{3}(v_s^{-3} - v_{s,1}^{-3}) - \frac{4}{5}K_{in}\ln\frac{v_s}{v_{in,1}} = K_{in}(\text{Fo} - \text{Fo}_1).$$

At $K_{in} \ll 1$ the relative temperature drop over the cross section does not exceed 0.5 K_{in} (Fig. 2) and the plate may be regarded as "thin." In this case the duration of the first stage is negligibly small, and the expression for the second stage differs from the exact solution only with respect to a very small logarithmic term (at $K_{in} = 5\cdot 10^{-2}$ and $v_s = 0.9$ the relative error does not exceed 3.3%, and at $v_s = 0.1$ it is less than 0.03%).

Heating (cooling) of a plate whose thermal coefficients depend on temperature. The skin effect is strongly expressed, p = const. The initial temperature distribution is uniform.

We assume some specific law of temperature dependence of λ and ρc. Let, for example,

$$\lambda = \lambda_0/\sqrt{1+\alpha T}, \tag{19}$$

$$\rho c = (\rho c)_0\left(\beta + \sqrt{1+\alpha T}\right). \tag{19a}$$

Substituting (19) in (1), we obtain

$$\vartheta = \frac{2}{\alpha}\left(\sqrt{1+\alpha T} - 1\right), \quad T = \frac{\lambda}{\lambda_0}\vartheta + \frac{\alpha}{4}\left(\frac{\lambda}{\lambda_0}\vartheta\right)^2. \tag{20}$$

Thus, from (19a) and (20) we have

$$\lambda = \lambda_0 \Big/ \left(1 + \frac{\alpha}{2\lambda_0}\lambda\vartheta\right),$$

$$\rho c = (\rho c)_0 \left(\beta + 1 + \frac{\alpha}{2\lambda_0}\lambda\vartheta\right). \qquad (21)$$

In the first stage the distribution of temperature (more exactly, the quantity $\lambda\vartheta$) over the plate cross section is described by equation (8). As in the previous example, to find the dependence on time we use the Kantorovich method in conjunction with the method of moments. We substitute (10a) and (11) in (6). In this case it is necessary to take into account that $\partial p/\partial\tau = 0$, but λ and ρc are known functions of $\lambda\vartheta$, which, in turn, depends on the x coordinate in accordance with (8).

After integrating (6) with respect to x, we obtain

$$\left\{\frac{7}{20}\,\frac{3+\lambda_0/\lambda}{1+\beta}\,\frac{\lambda_0}{\lambda_{in}}\,\frac{q}{R} + \frac{9}{280}\,\frac{\beta+2\lambda_0/\lambda_{in}}{1+\beta}\,\frac{pR\alpha}{\lambda_0}\left(\frac{q}{R}\right)^2 + \right.$$
$$\left. + \frac{11}{2688}\,\frac{1}{1+\beta}\left(\frac{pR\alpha}{\lambda_0}\right)^2\left(\frac{q}{R}\right)^3\right\}\frac{\partial}{\partial\tau}\left(\frac{q}{R}\right) = \frac{a_0}{R^2}.$$

Solving this differential equation with initial condition q = 0 at $\tau = 0$, we find

$$\frac{7}{40}\,\frac{\beta+\lambda_0/\lambda_{in}}{1+\beta}\,\frac{\lambda_0}{\lambda_{in}}\left(\frac{q}{R}\right)^2 + \frac{3}{280}\,\frac{\beta+2\lambda_0/\lambda_{in}}{1+\beta}\,\frac{pR\alpha}{\lambda_0}\left(\frac{q}{R}\right)^3 +$$
$$+ \frac{11}{10752}\,\frac{1}{1+\beta}\left(\frac{pR\alpha}{\lambda_0}\right)^2\left(\frac{q}{R}\right)^4 = \frac{a_0\tau}{R^2}. \qquad (22)$$

From (22) we determine the depth of the heating zone for any instant of time, then from (8) we calculate the corresponding value of $\widetilde{\lambda\vartheta}$ and, finally, from (20) the temperature T.

The end of the first stage $\tau = \tau_1$ is determined from (22) with q = R. In the second stage the temperature distribution is expressed by a parabola (9), and the function ζ_2 and the value of the operator $L(\widetilde{\lambda\vartheta})$ by expressions (10b) and (11a). Integrating (5) with a account for the temperature dependence of the thermal parameters, we obtain an ordinary differential equation for determining $(\widetilde{\lambda\vartheta})$ as a function of time. Solving it with the initial condition $(\lambda\vartheta)_c = (\lambda\vartheta)_{in}$ with $\tau = \tau_1$, we find

$$A\left[(\lambda\vartheta)_c - (\lambda\vartheta)_{in}\right] + B\left[(\lambda\vartheta)_c^2 - (\lambda\vartheta)_{in}^2\right] +$$
$$+ C\left[(\lambda\vartheta)_c^3 - (\lambda\vartheta)_{in}^3\right] = pR\,\frac{a(\tau-\tau_1)}{R^2}, \qquad (23)$$

where

$$A = 1 + \frac{3}{20}\,\frac{2+\beta}{1+\beta}\,\frac{pR\alpha}{\lambda_0} + \frac{3}{16}\,\frac{1}{1+\beta}\left(\frac{pR\alpha}{\lambda_0}\right)^2,$$

$$B = \frac{1}{4}\,\frac{\alpha}{\lambda_0}\left[\frac{2+\beta}{1+\beta} + \frac{6}{80}\,\frac{1}{1+\beta}\left(\frac{pR\alpha}{\lambda_0}\right)^2\right],\ C = \frac{1}{12}\,\frac{1}{1+\beta}\left(\frac{\alpha}{\lambda_0}\right)^2.$$

We now calculate the value of $(\widetilde{\lambda\vartheta})_c$ at any instant of time from (23), then that of $(\widetilde{\lambda\vartheta})$ from (9), and, finally, the temperature T from (20).

Equations (22) and (23) may be simplified in two special cases. If the heat capacity does not depend on temperature, then $\beta = \infty$. However, if the thermal conductivity is also a constant, then $\alpha = 0$, and the problem reduces to the special case 3 already considered (linear differential equation, linear boundary conditions of the second kind).

As an example in Fig. 3 we have shown the variation of the temperature of an Armco-iron plate. The thickness of the plate $2R = 40 \cdot 10^{-2}$ m, the energy flux density at the surface $p = 10^6$ W/m^2. The thermophysical parameters of Armco-iron vary within wide limits. At T = 300° K, λ = 70 W/m · deg, and $\rho c = 5.85 \cdot 10^6$ J/m^3 · deg.

From these values we find from (19a) the auxiliary coefficients for the given temperature range: $\alpha = 10^{-2}$ deg^{-1}, $\beta = -0.645$, λ_0 = 140 W/m · deg, and $(\rho c)_0 = 0.625 \cdot 10^6$ J/m^3 · deg, $a_0 = 2.24 \cdot 10^{-4}$ m^2/sec.

At constant flux density as the temperature increases heating slows down, but the drop over the cross section increases (as a result of the increase in heat capacity and decrease in heat conduction). Linearization of the problem leads to a serious discrepancy with the solution obtained. If, for example, constant values corresponding to the initial temperature are assigned to the thermophysical parameters, the heating rate proves to be too high over the entire interval (Fig. 3). However, if these values are made to correspond to the temperature at the end of heating, the rate will be correspondingly too low.

The Kantorovich method was originally developed for solving elliptic differential equations and, in particular, boundary value problems of the theory of elasticity. Since only part of the solution is selected a priori, it gave more accurate results than the usual direct methods. However, as shown in this article, the Kantorovich method is especially effective in solving hyperbolic equations, including nonlinear nonstationary problems of the theory of heat conduction. In fact in a limited region (with respect to the space coordinate) the solution of a boundary value problem can be obtained with the necessary accuracy by using direct methods and making a suitable choice of the form [5] and number of the coordinate functions, even though this requires a large volume of computation. However, in solving nonstationary problems this possibility is not open to us, because the region of variation of the time coordinate is unbounded. By using the Kantorovich method it is possible to find the exact law of temperature with time, admittedly not for the original problem, but for a somewhat modified one, in which the temperature distribution with respect to the space coordinates is known only approximately.

The basic idea of the Kantorovich method—reduction of the problem to the solution of an ordinary differential equation—has, in fact, already been applied in a number of studies in combination with various direct methods: a variational method [6, 7], the integral heat balance method [8–10], the Galerkin method, and the method of averaging functional corrections [11]. Occupying an intermediate position between the exact and approximate methods, it possesses great generality and can be successfully employed in solving various two- and three-dimensional nonstationary and nonlinear problems of the theory of heat conduction.

NOTATION

T is the temperature; ϑ is the temperature function; u and v are its relative values; τ is the time; x is the coordinate; p is the energy flux density; q is the depth of heated zone; 2R is the plate thickness; λ is the thermal conductivity; ρ is the density; c is the specific heat; γ is the coefficient of mutual irradiance; σ is the Stefan-Boltzmann constant; Fo is the Fourier number; K is the criterion analogous to the Biot number of linear problems. Subscripts: in is the initial; s is surface; c is central; m is medium; eq is equivalent; em is electromagnetic.

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31 January 1966